**TASK 16: PARTIAL FRACTIONS and PROOF BY MATHEMATICAL INDUCTION**

**Extended investigation**

**Unit 2**

**Topic 2.3: Real and complex numbers**

**Course-related information**

The skills and concepts developed in this extended investigation relate to the following dot points within the WA Mathematics Specialist syllabus:

2.3.4 develop the nature of inductive proof, including the ‘initial statement’ and inductive step

2.3.5 prove results for sums, such as  for any positive integer *n*

2.3.6 prove divisibility results, such as  is divisible by 5 for any positive integer *n*

2.3.7 define the imaginary number as a root of the equation 

2.3.8 represent complex numbers in the rectangular form;  where *a* and *b* are the real and imaginary parts

2.3.9 determine and use complex conjugates

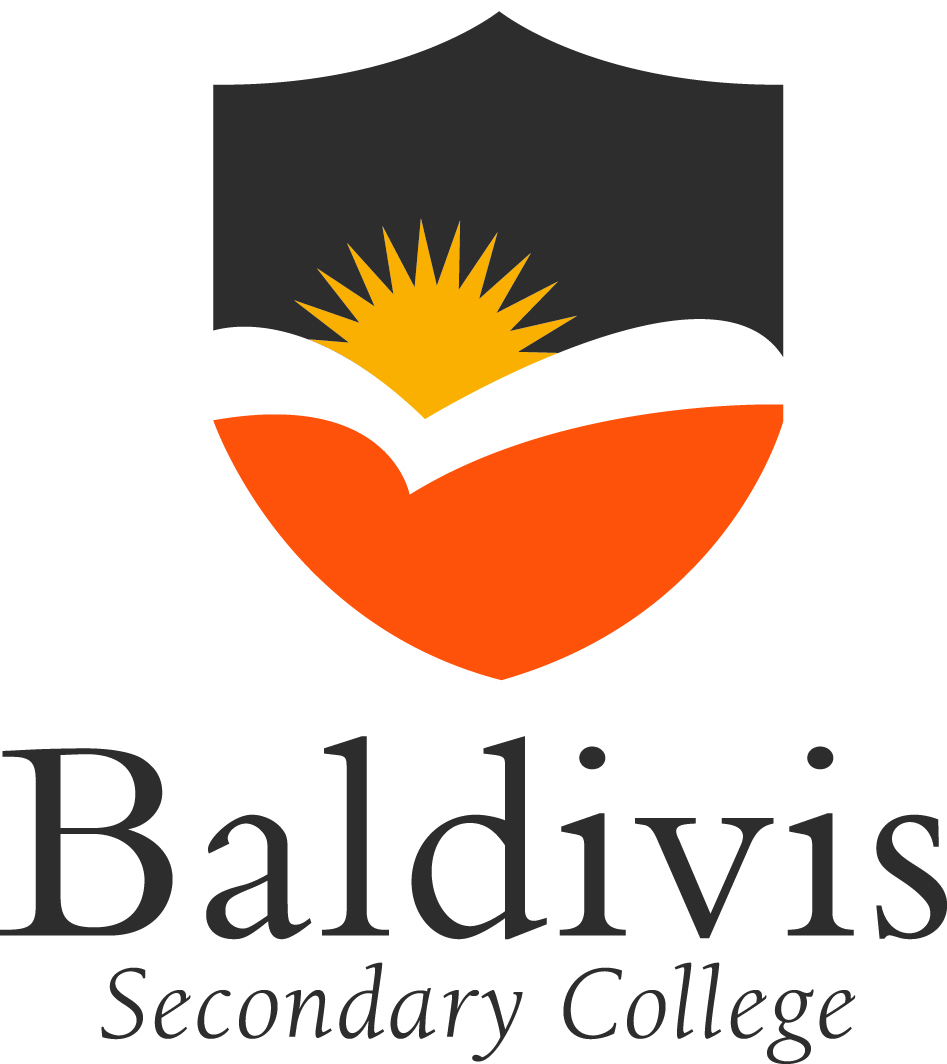
2.4.0 perform complex number arithmetic: addition, subtraction, multiplication and division

**Background information**

Students will develop the ability to express rational functions in terms of the sum of simpler fractions. They will also be introduced to the method of proof by mathematical induction. It is assumed that this method of proof will not be studied in class prior to the completion of this extended task. It is expected that students are able to factorise algebraic expressions for both real and complex factors, can perform simple arithmetic involving complex numbers and can add and subtract algebraic fractions with and without the use of CAS facilities.

**Task conditions**

It is expected that students will not have access to calculators for the in-class validation. No student notes are required.

 **Name\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

**Maths Specialist – Investigation 4 2016**

**PARTIAL FRACTIONS and PROOF BY MATHEMATICAL INDUCTION**

**Extended investigation Part 1:** **Preparation activity**

**Part A: Partial Fractions**

If *P* is a polynomial such that



where , then the degree of  is , i.e. .

The function  where  and  are polynomials is referred to as a rational function. It is possible to express  as a sum of simpler fractions, called ***partial fractions***, provided that the degree of  is less than the degree of . Such a rational function is called ***proper***.

If  is not proper, i.e. , then divide  into  until a remainder  is obtained such that . We now have



where  and  are also polynomials.

Thus

 is a *proper* rational function

 and  are *improper* rational functions.

To Rewrite Improper Rational Functions in terms of Proper Rational Functions

**-** rewrite the numerator in terms of the denominator until the remaining terms are of a lower degree than the denominator.

Example 1:

****

Check that 

Example 2:



Check that 

**Question 1**

### Rewrite the following improper algebraic fractions in terms of proper fractions.

### (a)

(b) 

(c) 

(d) 

To Write a Proper Rational Function as a Sum of Simpler Fractions (Partial Fractions):

### CASE I: the denominator is a product of distinct linear factors

i.e. we can write 

where no factor is repeated (and no factor is a constant multiplier of another). In this case, there exist constants  such that



The following example shows how these constants can be determined.

**Example**: 

Factorise the denominator .

Since the denominator has 2 distinct linear factors, the partial fraction decomposition has the form



To determine the values of  and , multiply both sides of the equation by the product of the denominators



The values of *A* and *B* may be found either by **substituting suitable values of *x*.**

Substituting values of *x* that make the coefficients of *x* equal to zero*,*

when *x = 2, -1 = A(-1)* hence *A = 1*

when *x = 3, 3 = B* hence *B = 3*

Hence, 

**CASE II:** the denominator **** is a product of linear factors, some of which are repeated.

Suppose the first linear factor  is repeated  times; that is  occurs in the factorisation of ****. Then instead of the single term , we would use



**Example**: 

Factorise the denominator 

Since the linear factor  occurs twice, the partial fraction decomposition is



Multiplying by the lowest common denominator



Substituting suitable values of *x,*

when *x = -1, 3 = 9A* hence *A = *

when *x = 2, 6 = 3C* hence *C = 2*

when *x = 0, 4 = 4A –2B + C* hence B = 

Hence, 

**CASE III:** the denominator**** contains irreducible quadratic factors, none of which is repeated.

If **** has the factor , where , then the expression for  will have a term of the form



where  and  are constants to be determined.

**Example**: 

Since cannot be factored further without using complex numbers, we write



Multiplying by  we have



Substituting suitable values of *x,*

when *x = -1, 6 = 2A* hence *A = 3*

when *x = 0, -1 = A + C* hence *C = -4*

when *x = 2, 15 = 5A + 6B + 3C* hence *B = 2*

Hence, 

**Example:** 

Since the degree of the denominator is *not less than* the degree of the numerator, we first rewrite the improper fraction in terms of proper fractions.



and



Multiplying both sides of this equation by 



Substituting suitable values of *x,*

when *x = 3, 18 = 7A* hence *A = *

when *x = -4, 4 = -7B* hence *B = *

Hence, 

**Question 2**

Express the following in partial fractions.

(a) 

(b) 

(c) 

(d) 

(e) 

**Part B: Proof by Mathematical Induction**

A proof by mathematical induction to show that statement  is true consists of three parts:

* Firstly, show that the result to be proved holds for a first case, usually (but not always) .
* Secondly, assume that the result is true for a certain value, i.e.  is true, and show that this leads to  being true.
* Finally, putting these two parts together, we can conclude that  is true for all values of *n* greater or equal to the first case.

**Example:** Use the method of proof by induction to prove that the sum of the first *n* odd integers is equal to.



 LHS = 1, RHS = hence, true for 

 Assume  is true, i.e.

 LHS , RHS 







= RHS

Hence, if  is true for , then  is true for  and  is true, thus, by induction,

**Example:** Use the method of proof by induction to prove  is always divisible by 5.

 a non-negative integer

 LHS = 0, which is divisible by 5 hence, true for 

 Assume  is true, i.e. a non-negative integer





which is divisible by 5.

Hence, if  is true for , then  is true for  and  is true, thus, by induction,

 is always divisible by 5.

**Question 3**

Prove the following statements using the method of proof by mathematical induction.

(a)

(b) is divisible by 5 is divisible by 5.

(c)

(d)

**PARTIAL FRACTIONS and PROOF BY MATHEMATICAL INDUCTION**

**Extended investigation Part 1:** **Preparation activity**

**Solutions**

**Question 1 (a)**

|  |
| --- |
|  |

**Question 1 (b)**

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**Question 1 (c)**

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**Question 1 (d)**

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**Question 2 (a)**

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**Question 2 (b)**

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|  |

**Question 2 (c)**

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| --- |
|  |

**Question 2 (d)**

|  |
| --- |
| Hence, |

**Question 2 (e)**

|  |
| --- |
| Hence, |

**Question 3 (a)**

|  |
| --- |
| LHS = 1, RHS  hence, true for  Assume  is true, i.e.    Hence, if  is true for , then  is true for  and  is true, thus, by induction, |

**Question 3 (b)**

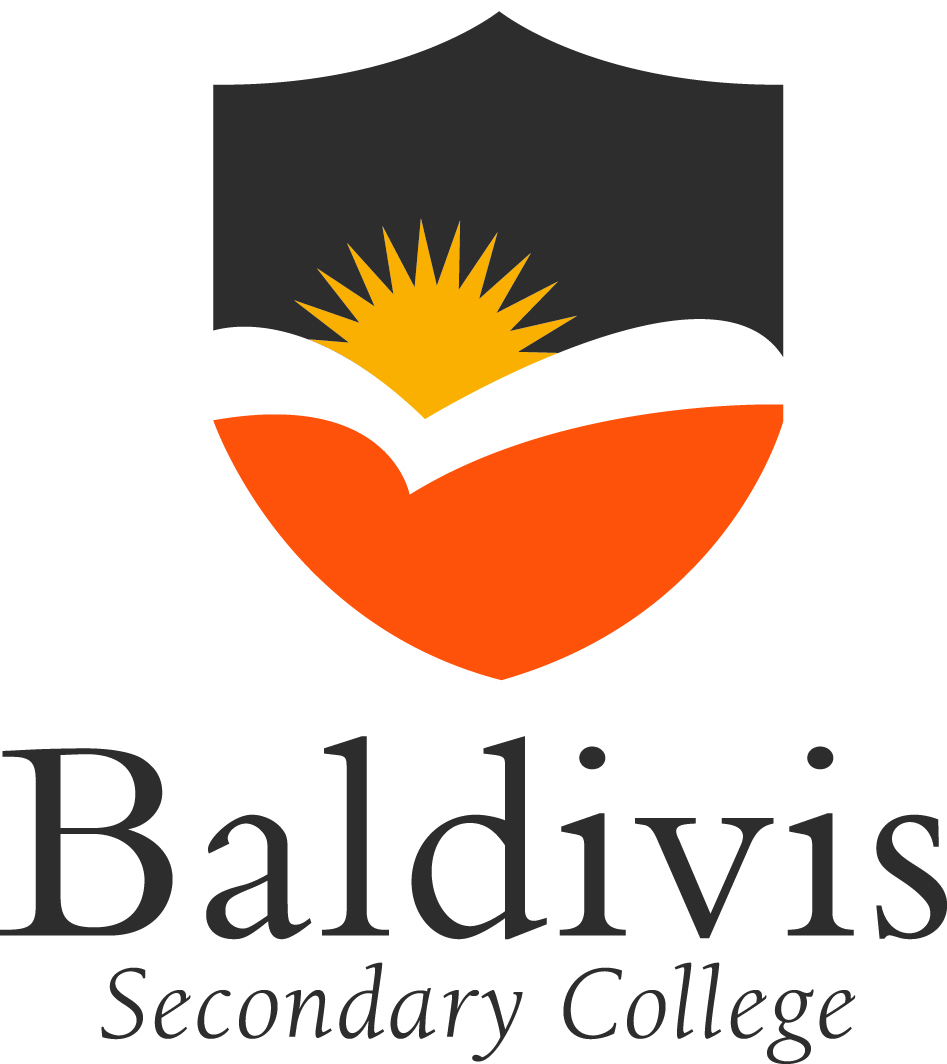
|  |
| --- |
| hence, true for  Assume  is true, i.e.      which is divisible by 5.  Hence, if  is true for , then  is true for  and  is true, thus, by induction,  is divisible by 5. |

**Question 3 (c)**

|  |
| --- |
| Since,  is true, assuming  is true for , and having proved is true for  then, by induction, |

**Question 3 (d)**

|  |
| --- |
| ince , and assuming is true for , having proved  is true for  and  thus, by induction, |

**Name\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_**

**Marks\_\_\_\_\_\_\_**

**PARTIAL FRACTIONS and PROOF BY MATHEMATICAL INDUCTION**

**Extended investigation Part 2:** **In-class validation**

**Question 1 (6 marks)**

### Rewrite the following improper algebraic fractions in terms of proper fractions.

(a)  (3)

(b)  (3)

**Question 2 (8 marks)**

(a) Express  in terms of partial fractions. (5)

(b) Hence show that . (3)

**Question 3 (6 marks)**

(a) Show that . (1)

(b) Hence, express in terms of three partial fractions. (5)

**Question 4 (5 marks)**

Prove

using the method of proof by mathematical induction.

**Question 5 (18 marks)**

(a) Express  in terms of partial fractions. (5)

(b) Suppose Then show that . (2)

(c) Suppose 

Prove

using the method of proof by mathematical induction. (5)

(d) 

Substitute  for *u* and  for *v* into  and simplify the resulting expression.

(2)

**PARTIAL FRACTIONS and PROOF BY MATHEMATICAL INDUCTION**

**Extended investigation Part 2:** **In-class validation**

**Solutions and marking key**

**Question 1 (a)**

|  |  |
| --- | --- |
|  | |
| Marking key/mathematical behaviours | Marks |
| * Rewrites numerator in terms of denominator * Correct quotient terms * Correct remainder | 1  1  1 |

**Question 1 (b)**

|  |  |
| --- | --- |
|  | |
| Marking key/mathematical behaviours | Marks |
| * Rewrites numerator in terms of denominator * Correct quotient terms * Correct remainder | 1  1  1 |

**Question 2 (a)**

|  |  |
| --- | --- |
|  | |
| Marking key/mathematical behaviours | Marks |
| * Factorises the denominator * Correctly expresses the fraction as a sum of two fractions * Correctly formulates an equation in terms of *A*, *B* and *x* * Determines *A* and *B* * Expresses the fraction in terms of partial fractions | 1  1  1  1  1 |

**Question 2 (b)**

|  |  |
| --- | --- |
| Using  from (a) | |
| Marking key/mathematical behaviours | Marks |
| * Uses (a) to rewrite the LHS * Recognises alternating signs of interior fractions * Simplifies resulting expression to establish RHS | 1  1  1 |

**Question 3 (a)**

|  |  |
| --- | --- |
| EITHER    OR | |
| Marking key/mathematical behaviours | Marks |
| * Establishes factors of cubic polynomial | 1 |

**Question 3 (b)**

|  |  |
| --- | --- |
|  | |
| Marking key/mathematical behaviours | Marks |
| * Correctly formulates an equation in terms of *A*, *B*, *C* and *x* * Determines *A* * Determines *B* * Determines *C* * Expresses the fraction in terms of partial fractions | 1  1  1  1  1 |

**Question 4**

|  |  |
| --- | --- |
| LHS = 1.4 = 4, RHS = hence, true for  Assume  is true, i.e.    RHS  LHS ,    = RHS  Hence, if  is true for , then  is true for  and  is true, thus, by induction, | |
| Marking key/mathematical behaviours | Marks |
| * Establishes  is true * Assumes  is true and states * Establishes * Concludes  is true | 1  1  2  1 |

**Question 5 (a)**

|  |  |
| --- | --- |
|  | |
| Marking key/mathematical behaviours | Marks |
| * Correctly expresses the fraction as a sum of three fractions * Correctly formulates an equation in terms of *A*, *B*, *C* and *x* * Determines *A* and *C* * Determines *B* * Expresses the fraction in terms of partial fractions | 1  1  1  1  1 |

**Question 5 (b)**

|  |  |
| --- | --- |
|  | |
| Marking key/mathematical behaviours | Marks |
| * Add fractions * Uses  to establish sum | 1  1 |

**Question 5 (c)**

|  |  |
| --- | --- |
| LHS = , RHS = from (b) hence, true for  Assume  is true, i.e.      Since  is true, and assuming is true for , and having proven  is true for  thus, by induction, | |
| Marking key/mathematical behaviours | Marks |
| * Establishes  is true * Assumes  is true and states * Establishes * Concludes  is true | 1  1  2  1 |

**Question 5 (d)**

|  |  |
| --- | --- |
| Substitute  for *u* and  for *v*: | |
| Marking key/mathematical behaviours | Marks |
| * Substitutes for * Simplifies | 1  1 |